

11/5/1a.

Talk based on Mrowka's PCMI lecture / Frenkel system.

Orbifold

an orbifold is a topological space X together with an atlas of orbifold charts (U_i, V_i, Γ_i) :

$$\cup U_i = X. \quad (\text{set-theoretically}).$$

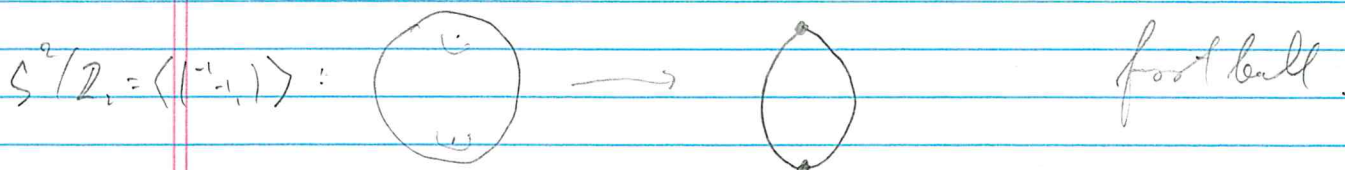
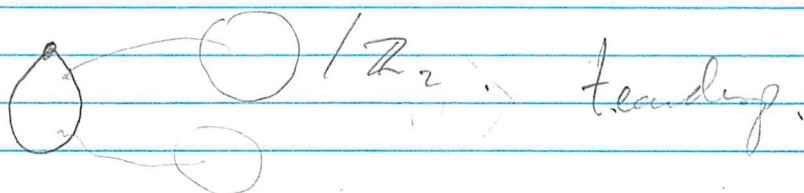
$$V_i \subset \mathbb{R}^n, \quad \Gamma_i \subset GL(\mathbb{R}^n) \text{ finite}$$

$$V_i / \Gamma_i \xrightarrow{\sim} U_i.$$

$$U_i \subset U_j \quad (\text{but only a cond'n for } U_i \subset U_j).$$

$$\Rightarrow V_i \subset V_j, \quad \Gamma_i \subset \Gamma_j$$

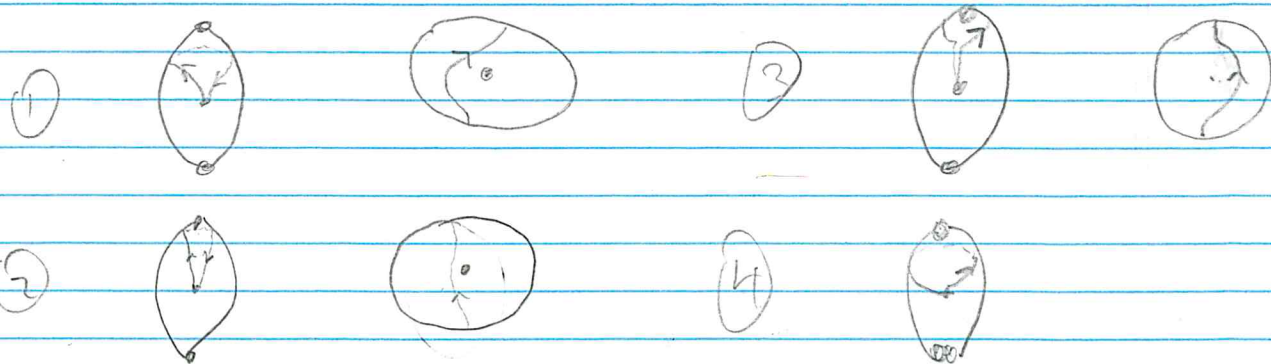
Examples:



Orbifold fundamental group

$$\pi_1^{\text{orb}} |X| = \{ \text{loops together w/ lifts to } V_i \} / \sim$$

Example: $\pi_1^{\text{orb}} (S^2 / \mathbb{Z}_2) = \mathbb{Z}_2.$

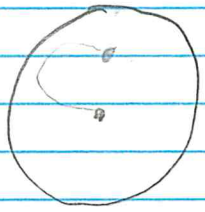


Can prove using "l.c.s." of an orbifold fibration

$$\mathbb{Z}_2 \rightarrow S^2 \rightarrow S^2/\mathbb{Z}_2$$

$$\pi_1(\mathbb{Z}_2) \rightarrow \pi_1(S^2) \rightarrow \pi_1^{orb}(S^2/\mathbb{Z}_2) \xrightarrow{\sim} \pi_0(\mathbb{Z}_2) \rightarrow \pi_0(S^2)$$

Make point that these paths do have lifting property



(want to use a point that's ~~not~~ free.)

Orbifold principal bundles:

$$\begin{array}{ccc} V \times G & \xrightarrow{(\pi)} & P \cup G \\ \downarrow \pi & & \downarrow \pi \\ V & \xrightarrow{\pi} & U \subset X \end{array}$$

G-action on P, not necessarily free.
but assume π -action on $V \times G$ is free, with

but $V \times G / \pi \xrightarrow{\sim} \pi^{-1}(U) \subset P$.

(and π intertwines G-action).

$\Rightarrow$ P is a manifold.

Have l.c.s.: $\pi_1(G) \rightarrow \pi_1(P) \rightarrow \pi_1^{orb}(X) \rightarrow \pi_0(G)$

Example: Seifert fiber spaces.

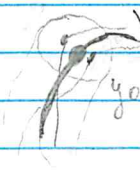
let $P = Y^3 \supset S^1$ s.t. $\text{stab}(y)$ finite.

$$X = Y^3 / S^1$$

$= \langle e \rangle$ generally

$V = \text{slice to } S^1\text{-orbit at } y_0$

Orbifold charts: $\text{stab}(y_0) = \mathbb{Z}_k$



$$\pi = \mathbb{Z}_k$$

$$V \times S^1 \rightarrow P$$

$$(x, m) \mapsto m(x)$$

○

$$\Sigma(p, q, r) = S^1 \cap \{z_1^p + z_2^q + z_3^r = 0\} \subset \mathbb{CP}^2.$$

p, q, r rel. prime.

$$u \in S^1: (z_1, z_2, z_3) \mapsto (u^{ar} z_1, u^{br} z_2, u^{cr} z_3)$$

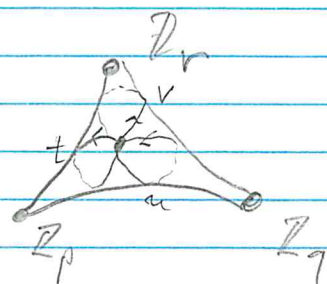
Self. foliated space:

$$\text{Stab}\{(0, z_2, z_3)\} = \mathbb{Z}_p \text{ etc.}$$

$$\text{Quotient is } S^2(p, q, r).$$

Why S^2 ?

$$(z_1, z_2, z_3) \mapsto [z_1^p, z_2^q] \in \mathbb{CP}^1.$$



○

$$V = B_z \times S^1$$

$$([v, 1], u) \mapsto \frac{(u^{ar} v, u^{br} \sqrt{1+v^p}, u^{cr})}{\sqrt{|v|^2 + |1+v^p|^{1/q} + 1}}$$

choose branch.

$$u^{par}(v^p - 1 - v^p + 1) = 0 \checkmark.$$

$$\pi_{orb}(S^2(p, q, r)) = \{t, u, v \mid t^p = u^q = v^r = 1, t u v = 1\}.$$

Notice $\text{Abel}(\pi_{orb}(S^2(p, q, r))) = \{e\}$: (triangle group).

$$1 = (tuv)^{pq} = t^{pq} u^{pq} v^{pq}, \text{ but } pq \text{ is invertible mod } r$$

$$\Rightarrow v = 1$$

$$f = u = 1 \text{ also. } \checkmark.$$

$$1 \rightarrow \pi_1(S^1) \rightarrow \pi_1(\Sigma(p, q, r)) \rightarrow \pi_1^b(S^2(p, q, r)) \rightarrow 1.$$

central.

$$\Rightarrow \pi_1(\Sigma(p, q, r)) = \{ \tilde{t}, \tilde{u}, \tilde{v}, h \text{ central} / \tilde{t}^p = h^{-1}, \text{ etc.} \}$$

Can show $\text{Abel}(\pi_1(\Sigma)) = \text{triv.}$

Representation variety:

$$\{ \rho : \pi_1(\Sigma(p, q, r)) \rightarrow \text{SU}(2) \}$$

$$\Rightarrow \rho(h) = \pm 1.$$

$$\xrightarrow{\sim} \{ \rho : \pi_1^{\text{orb}}(\Sigma^2(p, q, r)) \rightarrow \text{SO}(3) \}$$

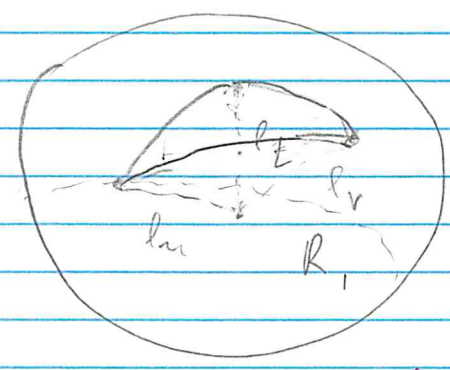
$$\text{Let } T = \rho(t), U = \rho(u), V = \rho(v).$$

$$T = \text{rotation by } 2\alpha = \frac{2\pi h}{p} \text{ about } l_t$$

$$U = \quad \quad \quad 2\beta = \frac{2\pi l}{q} \quad \quad \quad l_u$$

$$V = \quad \quad \quad 2\delta = \frac{2\pi m}{r} \quad \quad \quad l_v$$

ρ irred $\Rightarrow$ axes are not splines.



Claim: l_t, l_u, l_v lie on a

spherical triangle w/ angles α, β, δ . (exc. 2, 3, 6 case)

Write as products of reflections.

$$T = R_1 R_2 R_3, \quad R_1 \text{ refl. through plane } l_t$$

$$U = R_2 R_3 R_4, \quad R_2 \text{ " " " " } l_u$$

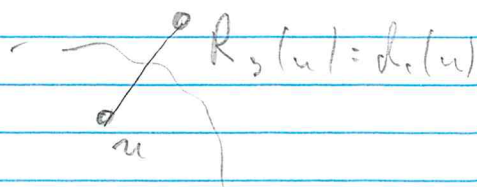
$$V = R_3 R_4 R_5, \quad R_3 \text{ " " " " } l_v$$

$$TUV = 1 = R_1 R_2 R_3 R_4 R_5 R_6 = R_1 R_2 R_3 R_4 R_5$$

$$\Rightarrow R_3 = R_1 \underbrace{R_2 R_{uv}}_{\text{fixes } u.}$$

$\downarrow$ $\swarrow$
 do not fix u .

$$\Rightarrow R_3(u) = R_1(u)$$



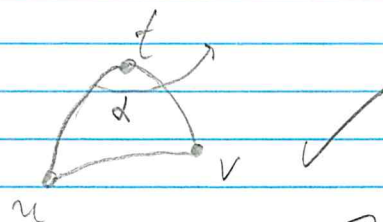
$$\Rightarrow R_3 = R_1$$

$\Rightarrow$ fixes both t and v

$$\Rightarrow R_1 = R_{vt} = R_3$$

$$\Rightarrow R_2 = R_{uv}$$

$$T = R_{vt} R_{ut}$$



Name 4 antipodal triangles



$\angle$ Angles $> \pi$

$$\alpha + \beta + \gamma > 1$$

$$1 - \alpha - 1 - \beta + \gamma > 1$$

$$\gamma - \alpha - \beta > -1$$

$$\alpha + \beta - \gamma < 1$$

$$\alpha - \beta + \gamma < 1$$

$$\alpha, \beta, \gamma < \pi$$

$$\alpha, \pi - \beta, \pi - \gamma$$

$$\pi - \alpha, \beta, \pi - \gamma$$

$$\pi - \alpha, \pi - \beta, \gamma$$

close

$$\beta + \gamma < \pi$$

s.t. d.m

same parity

$$\frac{k}{p} + \frac{l}{q} + \frac{m}{r} > 1$$

$$\frac{k}{p} + \frac{l}{q} + \frac{m}{r} < 1$$

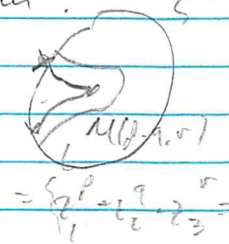
etc.

$\lambda(\Sigma(p, q, r)) = \text{count of these h.l.m.}$

OBSERVATION: $= -\frac{1}{4} \sigma(M(p, q, r))$

clear fact, Floor grad sy is even, so

these guys freely generate I_*



Ex: $\Sigma(2, 3, 5)$

$$\frac{h}{2} + \frac{l}{3} + \frac{m}{5} > 1$$

$(1, 1, 1), (1, 1, 3)$

$(1, 2, 2)$

$$\frac{h}{2} + \frac{l}{3} - \frac{m}{5} < 1$$

$(1, 2, 4)$

$$\frac{h}{2} - \frac{l}{3} + \frac{m}{5} < 1$$

$I_*(\Sigma(p, q, r)) = \mathbb{Z} \oplus \mathbb{Z}$

(Can also compute CS.)

Rep. spaces of knot groups:

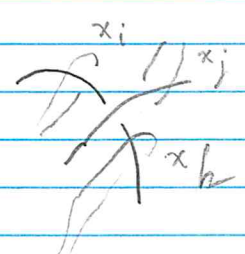
$$K \subset S^3$$

$$R_1(K) = \{p : \pi_1(S^3 \setminus K) \rightarrow SU(2) \mid p(\gamma)^2 = -1\}$$

$\forall \gamma \text{ meridional}$

$$\text{note: } p(\gamma)^2 = -1 \Leftrightarrow p \in S^2 \subset S^3$$

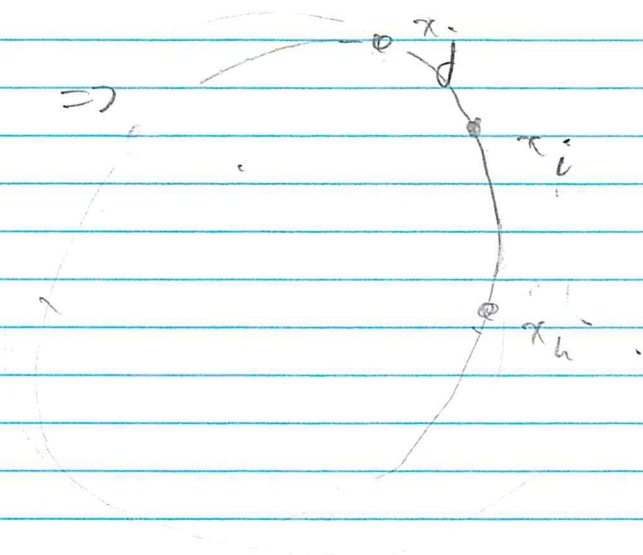
(analogous: $\{xI + yJ + zK \mid x^2 + y^2 + z^2 = 1\}$ in $\mathbb{H}$.)



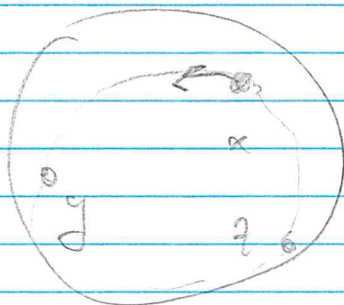
$$x_j \cdot x_i \cdot x_j^{-1} = x_k$$

Let $p(x_i)$ act on S^2 by conj'g.
 $\Rightarrow$ rot'n by π .

$$p(x_i) p(x_j | p(x_i))^{-1} = p(x_k)$$



Prefail:



$$R_1(S^3, T_{2,2}) = S^2 \sqcup SO(3)$$

$\mathbb{RP}^3$

$\mathbb{Z}_2 ? \mathbb{Z}$

$$R_1(S^3, T_{2,2m+1}) = S^2 \sqcup_m \mathbb{RP}^3$$

$$R_1(S^3, T_{2,2m+2}) = S^2 \sqcup_m S^1 \sqcup_m \mathbb{RP}^3$$

Obs. Herod. agrees w/ Khovanov!

$R_1(k) =$ prob. fold flat connections.

Need instanton herod. for these.